AVERAGED NORMS

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ABSTRACT

A method to construct an equivalent norm with both a rotundity and a smoothness property in a Banach space having two different equivalent norms, one with the rotundity and one with the smoothness property.

In a number of cases it has been proved that in some class of reflexive Banach spaces, one can in each space introduce an equivalent norm with some special property. For example, Kadec [4] has proved a result that each separable reflexive Banach space admits an equivalent locally uniformly rotund norm, and recently Lindenstrauss [5] has proved that each reflexive Banach space admits an equivalent rotund norm. In these cases, the classes are closed with respect to taking dual spaces, and then the theorems say that one may find other equivalent norms with the properies dual to those mentioned. A natural question is then whether one can find an equivalent norm satisfying both properties. We will show in this note a simple averaging procedure which yields an affirmative answer in the two cases mentioned above. In order to be applicable in other cases we will state the basic estimating lemma in a more general setting.

Let E be a (real) vector space and let f_0, g_0 denote convex functions on E, taking values in $\mathbb{R} \cup \{+\infty\}$ but not identically $+\infty$. Furthermore, we assume that the functions are homogeneous of second degree:

$$f_0(tx) = t^2 f_0(x),\ g_0(tx) = t^2 g_0(x) \text{ for all } t \text{ in } \mathbb{R} \text{ and } x \text{ in } E$$

hence nonnegative and vanishing at the origin of E. Finally, we assume that $_0$ and g_0 are equivalent in the sense that there exists a positive number C such that

$$g_0 \leqq f_0 \leqq (1 + C) g_0$$

Now we form the average of f_0 and g_0 in two different ways. The first is the ordinary average

$$f_1 = \frac{1}{2}(f_0 + g_0)$$

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The second is the "inf-convolution average" g_1 defined by

$$g_1(x) = \inf \left\{\frac{1}{2}(f_0(x+y) + g_0(x-y)) : y \in E\right\}$$

It is easy to see that f_1 and g_1 are convex, homogeneous of second degree, and that they satisfy the relations

$$\begin{aligned} g_0 \leqq g_1 &\leqq f_1 \leqq f_0 \\ f_1 &\leqq (1 + 2^{-1}C)g_1 \end{aligned}$$

Now iterate this procedure:

$$\text{(1)} \qquad \begin{aligned} f_{n+1}(x) &= \frac{1}{2}(f_n(x) + g_n(x)) \\ g_{n+1}(x) &= \inf \ \frac{1}{2}(f_n(x+y) + g_n(x-y)) : y \in E \end{aligned} \qquad \text{for } n \geqq 0$$

The result is two sequences of functions that satisfy the relations

$$\begin{aligned} g_n \leqq g_{n+1} \leqq f_{n+1} \leqq f_n \\ f_n \leqq (1 + 2^{-n}C)g_n \end{aligned} \qquad \text{for all } n \geqq 0$$

Hence the two sequences converge (uniformly on each set on which f_0 is bounded) to a likewise second degree homogeneous convex function h

$$h(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x)$$

$$\text{(2)} \qquad (1 + 2^{-n}C)^{-1}h \leqq g_n \leqq h \leqq f \leqq (1 + 2^{-n}C)h$$

The estimates (2) of the speed of this convergence are however too crude to be useful. The object of the main lemma is to improve these estimates.

LEMMA $g \leqq f \leqq (1 + 4^{-n}C)g_n$ *for all* $n \geqq 0$.

The proof is by recursion starting with $n = 0$. Assume n is the largest natural number for which the lemma has been proved. We have then the following estimates, using homogeneity and convexity and putting for the time being $1 + 4^{-n}C/2 = a$

$$f_{n+1} \leqq f_n, \ f_{n+1} \leqq a\, g_n,$$

$$\begin{aligned} \frac{1}{2}(f_n(x+y) + g_n(x-y)) &\geqq \frac{1}{2}\left(\frac{1}{a^2} f_{n+1}(ax + ay) + \frac{1}{a} f_{n+1}(x-y)\right) \\ &= \frac{1}{2}\,\frac{1+a}{a^2}\left(\frac{1}{1+a} f_{n+1}(ax+ay) + \frac{a}{1+a} f_{n+1}(x-y)\right) \\ &\geqq \frac{1+a}{2a^2} f_{n+1}\left(\frac{2ax}{1+a}\right) = \frac{2}{1+a} f_{n+1}(x) \end{aligned}$$

Taking the inf over y of both sides above, one obtains the desired conclusion:

$$f_{n+1}(x) \leqq (1 + 4^{-(n+1)}C)g_{n+1}(x)$$

Thus the lemma is proved.

From now on we will assume that the functions are finite-valued everywhere on E and vanish only at the origin.

We first prove that strict convexity of either of the starting functions f_0 or g_0 is inherited by h.

THEOREM 1. *If f_0 is strictly convex, then so is h.*

Proof. Because of the iteration procedure, one can write the function f_n as

$$f_n = \frac{1}{2^n} f_0 + h_n$$

where h_n is another convex function, Using the lemma we have

$$0 \leqq f_n - h \leqq f_n - g_n \leqq 4^{-n}Cg_n \leqq 4^{-n}Cf_0$$

Thus

$$\left(\frac{1}{2^n} - \frac{C}{4^n}\right)f_0 + h_n \leqq h \leqq \frac{1}{2^n}f_0 + h_n$$

Now suppose x and y are points in E such that $y \neq x$. We have the estimate

$$h(x) - 2h\left(\frac{x+y}{2}\right) + h(y) \tag{3}$$

$$\geqq \frac{1}{2^n}\left[f_0(x) - 2f_0\left(\frac{x+y}{2}\right) + f_0(y) - \frac{C}{2^n}(f_0(x) + f_0(y))\right]$$

Since f_0 is assumed to be strictly convex, the right hand side above is strictly positive for some n, proving that h is strictly convex, as asserted.

As indicated in the beginning of this paper the intended use of our averaging procedure was to work with on equivalent norms of a Banach space (or at least a normed space, but for our purposes here there is no loss of generality to assume it complete), so we will now suppose that E is a Banach space and that f_0 is related to one of the equivalent norms by the formula

$$f_0(x) = \frac{1}{2}\|x\|^2 \tag{4}$$

The reason to take this relationship is that the same one will then connect the associated dual norm in E^*, the conjugate Banach space, with the function f_0^* conjugate to f_0, which is defined by

$$f_0^*(x) = \sup\{\langle x, y\rangle - f_0(y) : y \in E\} \text{ for all } x \text{ in } E^*$$

The same remark applies to the other functions f_n, g_n and h — the related norms are all equivalent, in fact, they are given by the expressions

$$(2f_n(x))^{1/2},\ (2g_n(x))^{1/2} \text{ and } (2h(x))^{1/2}$$

respectively. Also, the conjugate functions f_n^*, g_n^* and h^* correspond in that way to equivalent norms of E^*. Moreover, the iteration relations (1) become inverted on the conjugate side:

$$f_{n+1}^*(x) = \inf\left\{\frac{1}{2}(f_n^*(x+y)+g_n^*(x-y)) : y \in E^*\right\}$$

$$g_{n+1}^*(x) = \frac{1}{2}(f_n^*(x)+g_n^*(x))$$

whereas the estimates related to the Lemma become

$$(1+4^{-n}C)^{-1}h^* \leqq f_n^* \leqq h^* \leqq g_n^* \leqq (p+4^{-n}C)h^*$$

All this follows from the general theory of conjugate functions in Banach spaces, for which we refer to Brøndsted [1], together with the elementary fact that if f is convex and homogeneous of second degree and C is a positive constant, then

$$(Cf)^* = C^{-1}f^*$$

Note that with respect to the general theory we are in a particularly simple case, since the functions involved are all everywhere continuous and a fortiori finite valued.

We are now ready to state the first main application.

THEOREM 2. *If in a Banach space E there exists one equivalent rotund norm and another equivalent norm whose dual is a rotund norm for E^*, then there exists a third equivalent norm with both these properties. In particular, each reflexive Banach space has an equivalent norm which is rotund and smooth.*

Proof. If $\|x\|$ is a rotund norm for E, then f_0 given by (4) is strictly convex. In the same way we may suppose that g_0^* is strictly convex on E^*. Therefore, by Theorem 1, h is strictly convex on E and h^* is strictly convex on E^*. Thus the norm related to h is rotund and its dual is rotund on E^*. By the result of Lindenstrauss referred to in the first paragraph, the hypotheses of the theorem are fulfilled if E is reflexive, in which case smoothness and rotundity are dual properties, so that the averaged norm will be both rotund and smooth.

Prof. Lindenstrauss has pointed out to the author that Theorem 2 settles the only remaining open question in the table of Day [2], in that it shows that $c_0(I)$ — for an arbitrary index set I — is scm, i.e. that it can be renormed with a norm which is both rotund and smooth. Actually, in [2] Day constructs an equivalent rotund norm for $c_0(I)$ and also an equivalent rotund norm for the dual space

$l_1(I)$ which is moreover a "conjugate norm", i.e. derives by duality from some norm on $c_0(I)$. Thus our averaging procedure applied to the two norms on $c_0(I)$ gives a third equivalent norm with both desired properties.

We will now investigate uniformity conditions on the strict convexity of the considered functions. A convex function f will be called *locally uniformly strictly convex* at x if the quantity

$$\inf\left\{f(x)-2f\left(\frac{x+y}{2}\right)+f(y):\|x-y\|\geqq\varepsilon\right\} \tag{5}$$

is strictly positive for all $\varepsilon>0$. Note that since all our norms are equivalent it does not matter which one is used for $\|y\|$ above. Also, supposing that f is one of our second degree homogeneous norm-related functions, we say that it is *uniformly strictly convex* if the quantity

$$\inf\left\{f(x)-2f\left(\frac{x+y}{2}\right)+f(y):\|x\|=1,\ \|x-y\|\geqq\varepsilon\right\}$$

is positive for all $\varepsilon>0$. We have then the following theorem

THEOREM 3 *If f_0 is (locally) uniformly strictly convex, then so is h.*

Proof. Take the appropriate inf of both sides of (3), then choose n large enough.

It is intuitively evident that any of the functions f_0, h, $g_0{}^*$, h^* is (locally) uniformly strictly convex if and only if the related norm is (locally) uniformly rotund. To check this in detail requires some quite elementary but messy computations which we outline here for the local case. Since we are free to choose our norm in (5), we take the one related to f as in (4) and express f in it. Then

$$\inf\left\{\frac{1}{2}\|x\|^2-\left\|\frac{x+y}{2}\right\|^2+\frac{1}{2}\|y\|^2:\|x-y\|\geqq\varepsilon\right\} \tag{6}$$

is strictly positive for all $\varepsilon>0$. By specialization to the case $\|x\|=\|y\|=1$, this implies local uniform rotundity of the norm, which is usually expressed as follows. For each u in E with $\|u\|=1$ there is a function $\delta(\varepsilon)$ with strictly positive values for $\varepsilon>0$, such that

$$\|v\|=\|u\|=1,\ \|u-v\|\geqq\varepsilon \text{ implies } 1-\left\|\frac{u+v}{2}\right\|>\delta \tag{7}$$

conversely, to see that (7) implies (6), we put $u=x/\|x\|$, $v=y/\|y\|$ but then we have to distunguish between three cases:

$$\text{I: } \big|\,\|x\|-\|y\|\,\big|\geqq\frac{\varepsilon}{2};\quad \text{II: } 0\leqq\|x\|-\|y\|\leqq\frac{\varepsilon}{2},\quad \text{III: } 0<\|y\|-\|x\|\leqq\frac{\varepsilon}{2}$$

Case I: $$\frac{1}{2}\|x\|^2-\left\|\frac{x+y}{2}\right\|^2+\frac{1}{2}\|y\|^2 \geqq \frac{1}{2}\|x\|^2-\frac{1}{4}(\|x\|+\|y\|)^2+\frac{1}{2}\|y\|^2$$
$$=\frac{1}{4}(\|x\|-\|y\|)^2 \geqq \frac{\varepsilon^2}{16}$$

Case II: Put $k=\|x\|/\|y\|$ and use $x+y=\|y\|(u+v+(k-1)u)$:
$$\frac{1}{2}\|x\|^2-\left\|\frac{x+y}{2}\right\|^2+\frac{1}{2}\|y\|^2 \geqq \|y\|^2\left(\frac{1+k^2}{2}-\left(\left\|\frac{u+v}{2}\right\|+\frac{k-1}{2}\right)^2\right)$$
$$=\|y\|^2\left(\left(\frac{1+k^2}{2}\right)^{1/2}+\frac{k-1}{2}+\left\|\frac{u+v}{2}\right\|\right)$$
$$\left(\left(\frac{1+k^2}{2}\right)^{1/2}-\frac{k-1}{2}-\left\|\frac{u+v}{2}\right\|\right)$$
$$\geqq \|y\|^2 k\left(1-\left\|\frac{u+v}{2}\right\|\right)=\|x\|\,\|y\|\left(1-\left\|\frac{u+v}{2}\right\|\right)$$

Case III: $$\frac{1}{2}\|x\|^2-\left\|\frac{x+y}{2}\right\|^2+\frac{1}{2}\|y\|^2 \geqq \|x\|\,\|y\|\left(1-\left\|\frac{u+v}{2}\right\|\right)$$

Since in case II and III one can estimate $\|u-v\|$:
$$\|u-v\|=\frac{1}{\|y\|}\left\|x-y-\frac{\|x\|-\|y\|}{\|x\|}x\right\| \geqq \frac{\varepsilon}{2\|y\|},\ \frac{\varepsilon}{2\|x\|}$$
it follows from (7) that (6) admits a strictly positive infimum. Thus f is locally uniformly strictly convex if and only if the related norm is locally uniformly rotund.

It follows that, as in Theorem 2, one can "mix" any of the three properties rotundity, locally uniform rotundity and uniform rotundity in E with any of the three in E^*. We leave it to the reader to imagine the details. Also, one can easily prove that other norm properties, like the "uniformly non-square" property invented by James [3] and related properties, are inherited from the norm related to f_0 to that related to h. Here we will only state the result on separable reflexive Banach space mentioned in the beginning.

THEOREM 4. *If E is a separable reflexive Banach space, then there exists an equivalent norm for E such that both it and its dual are locally uniformly rotund. Consequently, the norm and its dual are both Fréchet differentiable.*

Proof. The cited result of Kadec says that any separable Banach space has an equivalent locally uniformly rotund norm. For reflexive spaces, local uniform rotundity in one of the spaces implies Fréchet differentiability of the dual (but not conversely).

References

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